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Complex of three-dimensional solvable models

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Abstract
An overview is given of integrable models of quantum, classical and statistical mechanics defined as evolution models in wholly discrete $(2 + 1)$-dimensional space-time and based on a special type of auxiliary linear problem.

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1. Introduction

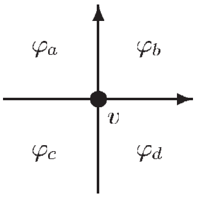
The systems we going to talk about are:

- Wholly discrete, i.e. lattice systems.
- Classical, quantum or spin systems.
- The systems may be associated with a discrete time evolution of auxiliary two-dimensional lattices, or with three-dimensional lattices.

In a two-dimensional world many discrete integrable systems may be formulated in two ways: firstly—as statistical mechanics systems, where one starts from the local Boltzmann weights on a 2D lattice, and secondly—as quantum mechanics systems, where one starts from a definition of a one-dimensional chain, while the second dimension—the discrete time—appears sometimes when one considers a kind of evolution of the chain.

The second approach is ours. A first step towards a three-dimensional integrable model in wholly discrete space–time is to look for an appropriate definition of a mapping, associated with two-dimensional auxiliary lattices. Such mappings would consist of three-dimensional models.

Below we will formulate several rules, allowing us to define some objects associated with two-dimensional auxiliary lattices. The main advantage of the method is that everything follows from just a set of linear equations. The objects we define will allow one to construct completely integrable models as $(2 + 1)$ quantized evolution systems. The quantization means



$$\Leftrightarrow \ \ell_v \stackrel{def}{=} \varphi_a + \varphi_b\, q^{1/2}\mathbf{u}_v + \varphi_c\, \mathbf{w}_v + \varphi_d\, \kappa_v \mathbf{u}_v \mathbf{w}_v \ = \ 0 \, .$$

Figure 1. The vth vertex.

that we will deal with the local Weyl algebra, which has a lot of well defined limits and thus we will obtain a lot of integrable systems.

The reader may find details concerning the systems considered in [15, 17–22].

2. Auxiliary linear problem

The main auxiliary object in our consideration is an arbitrary two-dimensional graph, formed by oriented straight lines. The elements of the graph are (1) vertices, (2) edges and (3) sites.

Consider an *oriented* vertex v, surrounded by four sites a, b, c, d as is shown in figure 1. To each vertex v we assign the local Weyl pair

$$v \ : \ \boldsymbol{u}_v \qquad \boldsymbol{w}_v \ : \ \boldsymbol{u}_v \boldsymbol{w}_v = q \ \boldsymbol{w}_v \boldsymbol{u}_v \tag{1}$$

$\boldsymbol{u}_v$ and $\boldsymbol{w}_v$ are supposed to be invertible. The Weyl elements for different vertices of a given graph commute. Also a $\mathbb{C}$ numerical parameter κ_v is assigned to the intersection of two lines, i.e. at the moment to the same vertex.

To each site s of the graph we assign *a site linear variable* φ_s, belonging to a formal left module of all vertex Weyl algebras. For the sites a, b, c, d in figure 1 their linear variables are φ_a, φ_b, φ_c and φ_d.

Four linear site variables, surrounding the vertex v, are to be combined into the *vertex local linear form*

$$\ell_v \stackrel{\text{def}}{=} \varphi_a + \varphi_b q^{1/2} \boldsymbol{u}_v + \varphi_c \boldsymbol{w}_v + \varphi_d \, \kappa_v \boldsymbol{u}_v \boldsymbol{w}_v . \tag{2}$$

The linear equation, corresponding to each vertex, is just the following equation for the linear site variables:

$$\ell_v = 0. \tag{3}$$

When a graph consists of many vertices, such an equation is to be written for each vertex. So, in general, for any graph it appears as a set of linear equations for a set of site variables. We will call the system of equations $\ell_v = 0$ the *complete linear system* of the graph.

3. *R*-mapping

The next step is to consider the 2D simplex: a triangle. Three lines may intersect in two ways—see figure 2. Consider at first the left triangle. There are three vertices, labelled by v_1, v_2 and v_3, and seven surrounding sites, whose linear variables are shown in figure 2. Three vertex linear equations in terms of the Weyl elements $\boldsymbol{u}_1$, $\boldsymbol{w}_1$ for the vertex v_1 and so on, defined by (2), (3) and figure 1, form the complete linear system for the left-hand side graph. Due to this system three linear variables may be expressed via four other ones.

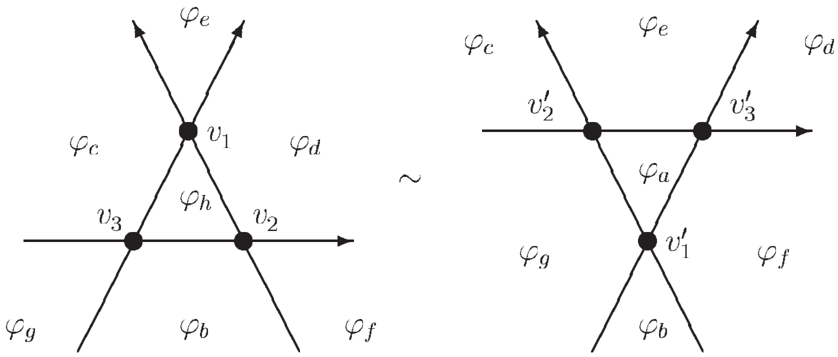


Figure 2. Two triangles.

Analogously one may write out the complete linear system for the right-hand side graph of figure 2. This system is to be written in the terms of Weyl elements $\boldsymbol{u}_1'$, $\boldsymbol{w}_1'$ for v_1', etc., but the κ parameters for v_1', v_2' and v_3' are to be the same as for v_1, v_2 and v_3.

The linear variables, surrounding the triangle, are the same in both graphs, except the internal ones—they are different. The integrability is to be based on the zero curvature condition. In our case this condition is simply the condition of linear equivalence of the complete linear systems of the left- and right-hand sides. In the considered case of the triangles, the left and right systems are to be equivalent after excluding the internal linear variables φ_h and φ_a.

Proposition 1. *The condition of linear equivalence of two triangles has the unique solution:*

$$\begin{array}{lll} \boldsymbol{w}_1' = \boldsymbol{w}_2 \cdot \Lambda_3 & \boldsymbol{w}_2' = \Lambda_3^{-1} \cdot \boldsymbol{w}_1 & \boldsymbol{w}_3' = \Lambda_2^{-1} \cdot \boldsymbol{u}_1^{-1} \\ \boldsymbol{u}_1' = \Lambda_2^{-1} \cdot \boldsymbol{w}_3^{-1} & \boldsymbol{u}_2' = \Lambda_1^{-1} \cdot \boldsymbol{u}_3 & \boldsymbol{u}_3' = \boldsymbol{u}_2 \cdot \Lambda_1 \end{array} \tag{4}$$

where

$$\begin{aligned} \Lambda_1 &= \boldsymbol{u}_1^{-1} \cdot \boldsymbol{u}_3 - q^{1/2}\, \boldsymbol{u}_1^{-1} \cdot \boldsymbol{w}_1 + \kappa_1\, \boldsymbol{w}_1 \cdot \boldsymbol{u}_2^{-1} \\ \Lambda_2 &= \frac{\kappa_1}{\kappa_2}\, \boldsymbol{u}_2^{-1} \cdot \boldsymbol{w}_3^{-1} + \frac{\kappa_3}{\kappa_2}\, \boldsymbol{u}_1^{-1} \cdot \boldsymbol{w}_2^{-1} - q^{-1/2}\, \frac{\kappa_1\, \kappa_3}{\kappa_2}\, \boldsymbol{u}_2^{-1} \cdot \boldsymbol{w}_2^{-1} \\ \Lambda_3 &= \boldsymbol{w}_1 \cdot \boldsymbol{w}_3^{-1} - q^{1/2}\, \boldsymbol{u}_3 \cdot \boldsymbol{w}_3^{-1} + \kappa_3\, \boldsymbol{w}_2^{-1} \cdot \boldsymbol{u}_3. \end{aligned} \tag{5}$$

Moreover, the mapping $\boldsymbol{u}_j, \boldsymbol{w}_j \mapsto \boldsymbol{u}_j', \boldsymbol{w}_j'$ is the canonical one with respect to the local Weyl algebra.

Now we consider the second important object of our exposition: the canonical invertible mapping $\boldsymbol{R}_{1,2,3}$, defined as

$$\boldsymbol{u}_j' = \boldsymbol{R}_{123}\boldsymbol{u}_j\boldsymbol{R}_{123}^{-1} \qquad \boldsymbol{w}_j' = \boldsymbol{R}_{123}\boldsymbol{w}_j\boldsymbol{R}_{123}^{-1} \qquad j = 1, 2, 3. \tag{6}$$

The question about an explicit realization of $\boldsymbol{R}_{123}$ will be discussed later.

Note that the origin of the mapping $\boldsymbol{R}$ resembles the method of the local Yang–Baxter equation (LYBE) [12]. LYBE just corresponds to another definition of the equivalence of two triangles. The reader may find a lot of information concerning several types of this equivalence in [7–9, 16].

By construction, $\boldsymbol{R}$ mapping concerns the three-dimensional models. This is provided by the triplet of Weyl algebras, and may be easily visualized if one puts two triangles of figure 2 one above the other and imagines $\boldsymbol{R}$ as a three legged cross formed by the lines from v_1 to v_1', from v_2 to v_2' and from v_3 to v_3'.

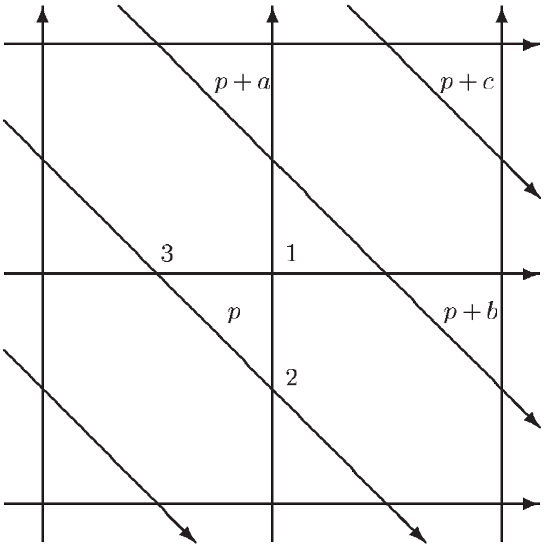


Figure 3. The Kagome lattice.

4. Auxiliary lattices

The concept of the equivalence of several graphs with the same outer structure allows one to derive unambiguously a mapping, corresponding to any re-gluing of a lattice. In any case, any complicated mapping may be decomposed into a set of primary mappings $\boldsymbol{R}$. The sequence of a decomposition is not essential, because any mapping is defined by the linear equivalence, and the equivalence condition has an unique solution—thus, all questions concerning, say, the tetrahedron equation do not arise at all.

A model may now be defined by the form and shape of the auxiliary lattice. Here we will deal mainly with the evolution on the Kagome lattice, but it is helpful to mention other lattices.

4.1. Kagome lattice

This is shown in figure 3. It is supposed that this lattice is drawn on a torus. Let p be an element of two-dimensional vector space, $\mathbb{Z}_A \times \mathbb{Z}_B$, spanned by vectors a and b ($c \equiv a + b$). We will use these vectors to mark out similar triangles of the Kagome lattice, see figure 3. It is implied that A and B are the spatial sizes of the lattice. Primary dynamical variables of the system are the set[1] of

$$\left\{\boldsymbol{u}_{j,p}, \boldsymbol{w}_{j,p}\right\} \qquad j = 1, 2, 3 \quad p \in \mathbb{Z}_A \times \mathbb{Z}_B. \tag{7}$$

Indices $j = 1, 2, 3$ marks the vertices of the pth triangle, see figure 3 again. The one step evolution mapping corresponds to the simultaneous shift of all inclined lines to the north-east through the vertices of the type 1. After such a shift the structure of the lattice is restored completely, especially in the case when the κ parameters depend only on a type of vertex, 1,

[1] We shall use the notation {, } to denote a set of something.

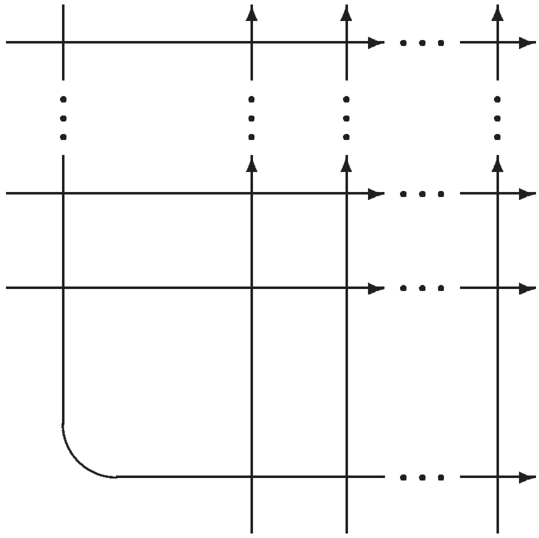


Figure 4. The square lattice.

2 or 3. One step evolution is the mapping $\boldsymbol{U}$, defined by

$$\begin{aligned} &\boldsymbol{U}\boldsymbol{u}_{1,p}\boldsymbol{U}^{-1} = \left(\boldsymbol{u}_1'\right)_p && \boldsymbol{U}\boldsymbol{w}_{1,p}\boldsymbol{U}^{-1} = \left(\boldsymbol{w}_1'\right)_p \\ &\boldsymbol{U}\boldsymbol{u}_{2,p}\boldsymbol{U}^{-1} = \left(\boldsymbol{u}_2'\right)_{p-a} && \boldsymbol{U}\boldsymbol{w}_{2,p}\boldsymbol{U}^{-1} = \left(\boldsymbol{w}_2'\right)_{p-a} \\ &\boldsymbol{U}\boldsymbol{u}_{3,p}\boldsymbol{U}^{-1} = \left(\boldsymbol{u}_3'\right)_{p-b} && \boldsymbol{U}\boldsymbol{w}_{3,p}\boldsymbol{U}^{-1} = \left(\boldsymbol{w}_3'\right)_{p-b}. \end{aligned} \tag{8}$$

Primed operators are given by the local formulae (4) and (5), and subscripts p, $p-a$ or $p-b$ are to be added to each $\boldsymbol{u}_j$, $\boldsymbol{w}_j$ of (4) and (5). Note again that $\kappa_{j,p} = \kappa_j$. The discrete time may now be defined simply as

$$\boldsymbol{u}_{j,p,t} = \boldsymbol{U}^t\, \boldsymbol{u}_{j,p}\, \boldsymbol{U}^{-t} \qquad \boldsymbol{w}_{j,p,t} = \boldsymbol{U}^t\, \boldsymbol{w}_{j,p}\, \boldsymbol{U}^{-t}. \tag{9}$$

Geometrically, the Kagome lattice appears as the section of the simple cubic lattice produced by a completely inclined plane. The one step evolution operator is formed by the vertices of the cubic lattice (i.e. by $\boldsymbol{R}$), situated between two adjacent inclined planes.

4.2. Other auxiliary lattices

Of all the other possible auxiliary lattices we will mention only two. The first one is the square lattice with one additional line, as is shown in figure 4. The square is the upper right part of figure 4. For this lattice we do not impose a toroidal boundary condition at the moment. The re-gluing is the pass of the additional line (this curved line is the left lowermost one) through all the vertices of the square part (so that it will become the right uppermost curved line). The corresponding mapping is a kind of monodromy operator. Taking a trace of the monodromy operator with respect to the Weyl algebras, corresponding to all vertices of the additional line, one obtains an object, defined on the square lattice with the toroidal boundary conditions. This traced operator is usually called the 'quantum transfer matrix', or 'Q operator'. The two spatial sizes of the square lattice are arbitrary. Let them be A and B again. One may show that the quantum transfer matrix arising after a small modification (connected with the transmutation of a dimension to a rank) is related to an integrable $U_q(\widehat{sl_A})$ chain of length B. In part, when $A = 2$, it appears to be the massive sine-Gordon model for arbitrary q, or the N-state integrable

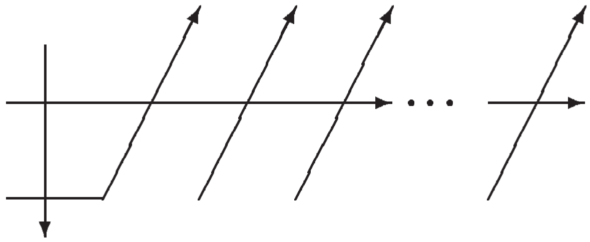


Figure 5. The 'spiral' lattice.

chiral Potts model when $q^N = 1$. For arbitrary A and B, and for $q^N = 1$ and some other restrictions (we will discuss later), this quantum transfer matrix is the layer-to-layer transfer matrix for the Zamolodchikov–Bazhanov–Baxter (ZBB) model [1, 2, 14].

The other auxiliary lattice to be mentioned is shown in figure 5. Here it is implied that the cyclic boundary condition in the bottom to top direction. This means the inclined segments form one spiral, and the horizontal line intersects this spiral in M points—this M is the size of the chain. The additional left line also implies the cylinder boundary condition. The mapping, corresponding to the pass of the additional line from left to right, is also some monodromy operator. Taking the trace over the Weyl content of two separated vertices, one gets a 'quantum transfer matrix'. One may show that this operator is connected with the Q operator for the relativistic Toda chain [13].

We give these two examples just to show the variety of all the possible auxiliary two-dimensional lattices.

5. The determinant

So, we have understood the richness of the variety of possible auxiliary lattices and the richness of the mappings. A natural question arisen is: why we are considering them?

The answer is: all these mappings are integrable in the usual sense. In all cases, dealing with an auxiliary lattice with the toroidal boundary conditions, formed by Δ vertices (and so by Δ sites—Euler's theorem for the torus), one may point out exactly $(\Delta + 1)$ independent operator-valued polynomials of $\{\boldsymbol{u}_v, \boldsymbol{w}_v\}$, invariant with respect to the evolution operator or with respect to a 'quantum transfer matrix'. The number of the integrals of motion is larger than the dimension of the phase space—this just means the existence of a mass centre.

All the mappings appear as the linear equivalence condition of two sets of linear equations. It is obvious intuitively: if a system of linear equations may be solved, then any equivalent system may also be solved (even if the coefficients are not $\mathbb{C}$ numbers). This means that the determinant of the complete linear system is to be an ideal of the mappings.

Consider the complete linear system for any lattice with the toroidal boundary conditions[2]. Due to linearity the toroidal boundary conditions for the linear variables φ_s may be written as the *quasiperiodic* boundary conditions:

$$\varphi_{p+Aa} = \alpha\varphi_p \qquad \varphi_{p+Bb} = \beta\varphi_p \qquad \alpha, \beta \in \mathbb{C} \tag{10}$$

for any φ. Furthermore, because the noncommutative Weyl elements only meet in the same ℓ_v, the operator-valued determinant of the complete linear system is well defined. Due to (10)

[2] The reader should keep in mind the case of the Kagome lattice. Moreover, the reader may test all these for the case of one triangle, $A = B = 1$, where this case corresponds to one site evolution operator. Note the appearance of a Hamiltonian of the isolated $\boldsymbol{R}$.

it depends on α, β:

$$\boldsymbol{J}(\alpha,\beta) = \det\ \{\ell_v\}\ \cdot\ \mathcal{N}^{-1}. \tag{11}$$

This normalization factor $\mathcal{N}$ is to be chosen so that in the decomposition

$$\boldsymbol{J}(\alpha,\beta) = \sum_{a,b} \alpha^a\ \beta^b\ \boldsymbol{J}_{a,b} \tag{12}$$

the element $\boldsymbol{J}_{0,0}\ \equiv\ 1$. In general $\mathcal{N}$ may be a monomial over α, β—it depends on the form in which the linear system has been written. Usually we make $\boldsymbol{J}_{a,b}$ equal to unity, corresponding to a corner of the Newton diagram of $\boldsymbol{J}(\alpha,\beta)$. There exists a nice combinatorial representation of the generation function, such that $\boldsymbol{J}_{a,b}$ corresponds to special paths on the auxiliary lattice with the homotopy class fixed, see [18, 19] for the details. Thus, the following propositions hold:

Proposition 2. *$\boldsymbol{J}(\alpha,\beta)$ as the function of α, β is invariant with respect to the quantum mapping operator (evolution operator $\boldsymbol{U}$ for the Kagome lattice as well as Q matrices for the other lattices).*

Proposition 3. *The set of $\{\boldsymbol{J}_{a,b}\}$ contains exactly $\Delta + 1$ independent integrals of motion.*

Proposition 4. *The set of $\{\boldsymbol{J}_{a,b}\}$ obeys*

$$\boldsymbol{J}_{a,b}\ \cdot\ \boldsymbol{J}_{a',b'} = q^{ab'-ba'}\ \boldsymbol{J}_{a',b'}\ \cdot\ \boldsymbol{J}_{a,b}. \tag{13}$$

It follows from the last proposition that $\boldsymbol{J}_{0,1}$ and $\boldsymbol{J}_{1,0}$ may be interpreted as the mass centre elements.

6. Well defined regimes

It is not necessary to explain all the disadvantages of the rational mappings of the local Weyl algebra here. To get the well defined quantum objects, such as $\boldsymbol{R}$ or $\boldsymbol{U}$, or get a good notion of the trace, one needs to consider several regimes and specifications of the formal Weyl algebra.

6.1. Classical model

The first and most obvious one is the limit when $q^{1/2} = -1$. In this case $\boldsymbol{u}_v$ and $\boldsymbol{w}_v$ become the usual classical canonical variables u_v and w_v with the Poisson braces following from (1):

$$\{\,u_v\,,\ w_v\,\}_P = u_v\, w_v. \tag{14}$$

We have used the subscript P to distinguish between the notation for a set and the notation for the Poisson braces. The formulae (8) are a kind of Hamilton equations of motion of the 'first order in time'. The generating function (11) defines the spectral curve $J(\alpha,\beta) = 0$. The Lagrange equations of motion are more suitable for the classical models. To derive them, one has to change the frame of references at first. Let the three-dimensional coordinate

$$\boldsymbol{p} = p_1\boldsymbol{e}_1 + p_2\boldsymbol{e}_2 + p_3\boldsymbol{e}_3 \tag{15}$$

be identified with (p,t) as

$$\boldsymbol{p} \equiv (p = p_2 a + p_3 b\,,\ t = p_1 + p_2 + p_3). \tag{16}$$

In these coordinates the Legendre transformation is

$$\begin{array}{lll} w_{1,\boldsymbol{p}} \sim \dfrac{\tau_{3,\boldsymbol{p}+\boldsymbol{e}_2}}{\tau_{3,\boldsymbol{p}}} & w_{2,\boldsymbol{p}} \sim \dfrac{\tau_{3,\boldsymbol{p}}}{\tau_{3,\boldsymbol{p}+\boldsymbol{e}_1}} & w_{3,\boldsymbol{p}} \sim \dfrac{\tau_{2,\boldsymbol{p}}}{\tau_{2,\boldsymbol{p}+\boldsymbol{e}_1}} \\ u_{1,\boldsymbol{p}} \sim \dfrac{\tau_{2,\boldsymbol{p}}}{\tau_{2,\boldsymbol{p}+\boldsymbol{e}_3}} & u_{2,\boldsymbol{p}} \sim \dfrac{\tau_{1,\boldsymbol{p}}}{\tau_{1,\boldsymbol{p}+\boldsymbol{e}_3}} & u_{3,\boldsymbol{p}} \sim \dfrac{\tau_{1,\boldsymbol{p}+\boldsymbol{e}_2}}{\tau_{1,\boldsymbol{p}}}. \end{array} \tag{17}$$

The signs '$\sim$' imply the existence of pre-exponents in $\tau_{\alpha,\boldsymbol{p}}$. The Lagrangian equations of motion are the set of three local equations:

$$r_1 \tau_{1,\boldsymbol{p}+e_2+e_3} \tau_{2,\boldsymbol{p}} \tau_{3,\boldsymbol{p}} = \tau_{1,\boldsymbol{p}} \tau_{2,\boldsymbol{p}+e_3} \tau_{3,\boldsymbol{p}+e_2} + s_2 \tau_{1,\boldsymbol{p}+e_2} \tau_{2,\boldsymbol{p}+e_3} \tau_{3,\boldsymbol{p}} + s_3^{-1} \tau_{1,\boldsymbol{p}+e_3} \tau_{2,\boldsymbol{p}} \tau_{3,\boldsymbol{p}+e_2} \tag{18}$$

and two other equations may be obtained by the cyclic permutation of the indices 1, 2, 3. The coefficients r_α and s_α depend on the pre-exponents previously mentioned. They may be chosen so that $r_1 = 1 + s_2 + s_3^{-1}$, etc., so that $\tau_{\alpha,\boldsymbol{p}} = 1$ solves (18). Equations (18) may be reduced to the Hirota equation [6] in the limit $\kappa_1 \ll \kappa_2 = \kappa_3 \ll 1$. An interesting feature of (18) is that they are form-invariant with respect to the cubic group. The permutations of 1, 2, 3 are trivial, while the reflections act highly nontrivially.

For a given size of the auxiliary lattice $\tau_{\alpha,\boldsymbol{p}}$ is in general a theta function on the Jacobian of the spectral curve, so that all the equations (18) hold due to the Fay identity, see [20] for details. In the infinite volume (18) has the solitonic solutions. Let

$$W_\alpha(\boldsymbol{p}) = \lambda_\alpha \mathrm{e}^{\mathrm{i}(k_1 p_1 + k_2 p_2 + k_3 p_3)} \tag{19}$$

where

$$\mathrm{e}^{\mathrm{i}k_1} = \frac{\lambda_1((\lambda_1 - \lambda_3) + s_1(\lambda_1 - \lambda_2))}{\lambda_2(\lambda_1 - \lambda_3) + s_1 \lambda_3 (\lambda_1 - \lambda_2)} \quad \text{and cyclic permutations.} \tag{20}$$

Next let

$$d(\lambda, \lambda') = \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1' & \lambda_2' & \lambda_3' \end{vmatrix} \qquad D(\lambda, \lambda') = \frac{d(\lambda, \lambda')\, d(\lambda^{-1}, \lambda'^{-1})}{d(\lambda^{-1}, \lambda') d(\lambda, \lambda'^{-1})}. \tag{21}$$

Then, as an example, a two-solitonic solution is

$$\tau_{\alpha,\boldsymbol{p}} = 1 + W_\alpha(\boldsymbol{p}) + W_\alpha'(\boldsymbol{p}) + D(\lambda, \lambda') W_\alpha(\boldsymbol{p}) W_\alpha'(\boldsymbol{p}). \tag{22}$$

For a special class of the initial data the classical evolution (8) may have a simple attractor, i.e. a point in the space of the dynamical variables[3] such that

$$u_{j,p,t} = u_{j,p,t+1} \qquad w_{j,p,t} = w_{j,p,t+1}. \tag{23}$$

Besides the simple attractor there exist several periodic attractors, corresponding to an incomplete factorization of the spectral curve.

6.2. *Toda chain-type models*

Formulae (4) and (5) admit the limit when $\boldsymbol{u}_j \mapsto 1$ when $q = 1$ and some special values of κ_j are chosen. $\boldsymbol{w}_j'$ are then expressed via $\boldsymbol{w}_j$ only. Consider now the first order of $\hbar$ in $q = 1 - \hbar$, $\boldsymbol{u}_j = 1 - \mathrm{i}\hbar \boldsymbol{p}_j$. Equations (4) and (5) produce some linear mapping $\boldsymbol{p}_j \mapsto \boldsymbol{p}_j'$. The quantum operators for such mappings are well defined in terms of $\boldsymbol{p}_j$ and $\boldsymbol{w}_j = \boldsymbol{e}^{x_j}$, $[x_j, \boldsymbol{p}_j] = i$. The corresponding models are very close to the Toda chain. This procedure, being applied to the auxiliary lattice in figure 5, gives the Toda chain exactly.

6.3. *Free bosonic model*

This is another kind of quasiclassical limit. Consider mapping $\boldsymbol{R}$ in the classical case, and consider the linear mapping of the differentials

$$\left\{ \boldsymbol{a} = \frac{\mathrm{d}\, u_j}{u_j}, \boldsymbol{a}^+ = \frac{\mathrm{d}\, w_j}{w_j} \right\} \mapsto \left\{ \boldsymbol{a}' = \frac{\mathrm{d}\, u_j'}{u_j'}, \boldsymbol{a}'^+ = \frac{\mathrm{d}\, w_j'}{w_j'} \right\}. \tag{24}$$

[3] This point corresponds to a degeneration of the spectral curve into the rational variety, so that the theta functions become the solitonic expressions. The attractor appears in the limit $t \mapsto +\infty$ when $\mathbb{I}k_1 > 0$.

$\boldsymbol{a}'$ and $\boldsymbol{a}'^{+}$ are linear functions of $\boldsymbol{a}$ and $\boldsymbol{a}'^{+}$. Equation (1) provides that this linear mapping conserves

$$\left[\boldsymbol{a}_j, \boldsymbol{a}_j^{+}\right] = 1. \tag{25}$$

Being applied to the complete Kagome lattice, this scenario produces the free bosonic model, evolving in the classical field

$$\left\{u_{j,p,t}, w_{j,p,t}\right\}. \tag{26}$$

The free bosonic model is the simplest bundle over the classical evolution model: the tangent one. At the attractor (23) the model simplifies significantly: it becomes Bazhanov's free bosonic model with Baxter's partition function [2, 10]. Preliminary calculations show that, for cyclic attractors, the partition function is off-critical (this is the subject of a separate investigation). In general, the free bosonic models is the best proving ground for several statistical mechanics predictions.

6.4. Strongly coupling regime

This regime is a pure quantum mechanical one, see [11]. It is connected with Faddeev's dualization [4, 5]. In brief, there exists a universal method to make the evolution operator $\boldsymbol{U}$ unitary and able to well define the Hilbert space, etc. To do this, one has to consider the pair $\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{w}}$ besides $\boldsymbol{u}, \boldsymbol{w}$, such that $q = e^{i\pi\tau}$ and $\tilde{q} = e^{-i\pi/\tau}$, and when $\tau = e^{i\theta}$ the dual Weyl pair is the conjugated primary one. It is expected that all the operators of the dualized mappings are well defined as are any of their products, the notion of the trace, and so on. This statement, however, is to be proved in a separate investigation.

6.5. Root of unity

The most interesting regime is $q^N = 1$. In this case the Weyl algebra has the natural centres $\boldsymbol{u}^N$ and $\boldsymbol{w}^N$. When they are numbers, the N-dimensional representation of the Weyl algebra arises. It is interesting to check, due to (4) and (5), that $\boldsymbol{u}_j'^N$ and $\boldsymbol{w}_j'^N$ are expressed via $\boldsymbol{u}_j^N$, $\boldsymbol{w}_j^N$ and κ_j^N by the classical formulae. Thus any mapping splits into two parts [3]: the first one is a matrix part, responsible for a changing of the matrix structure of $\boldsymbol{u}_v$, $\boldsymbol{w}_v$, and the second one is the classical part, responsible for the changing of the centres $\boldsymbol{u}_v^N$, $\boldsymbol{w}_v^N$. Thus one may talk about another kind of a bundle: the base is the set of $\boldsymbol{u}_v^N$, $\boldsymbol{w}_v^N$, while the typical layer is an element of N^{Δ}-dimensional Hilbert space (Δ is the number of vertices of the auxiliary lattice). Note that the observation concerning the classical dynamics of Nth powers was made first by Bazhanov and Reshetikhin [25].

The classical dynamics is known completely. In the attractor (23) the classical part of the evolution mapping trivializes, and it rests only on the finite-dimensional part. It appears exactly as the evolution operator for the ZBB spin model, and the matrix elements of $\boldsymbol{R}$ are given by the R matrix of the ZBB model, see [14]. Attractor conditions (23) at Nth powers are responsible for the high type curves in ZBB, the chiral Potts model, and so on.

Our method provides an universal functional equation for all the models at root of unity, no matter—on an attractor or in general position, because of the functional equation concerning the generating function $\boldsymbol{J}(\alpha, \beta)$. To derive it, look back at the formulation of the linear problem, equations (2), (3) and (11). For the classical part of the spin system, $\boldsymbol{u}_v^N$ and $\boldsymbol{w}_v^N$, it also exists as its own linear problem. Fix it as follows:

$$L_v \stackrel{\text{def}}{=} \Phi_a - \Phi_b \varepsilon_N \boldsymbol{u}_v^N + \Phi_c \varepsilon_N \boldsymbol{w}_v^N + \Phi_d \kappa_v^N \boldsymbol{u}_v^N \boldsymbol{w}_v^N = 0 \tag{27}$$

where $\varepsilon_N = (-)^{N-1}$. The boundary conditions for Φ are $\Phi_{p+Aa} = \alpha^N \Phi_p$ and $\Phi_{p+Bb} = \beta^N \Phi_p$: compare with (10) for the Kagome lattice. Analogously to (11) we have calculated the determinant $\mathcal{J}(\alpha^N, \beta^N)$ of the complete linear system of (27)—it is important to make the same $\mathcal{J}_{a,b} = 1$ as previously, see the note right after equation (12).

Proposition 5. *For any auxiliary lattice with the toroidal boundary conditions*

$$\det \boldsymbol{J}(\alpha, \beta) = \left(\mathcal{J}(\alpha^N, \beta^N)\right)^{N^{\Delta-1}} \tag{28}$$

where Δ *is the number of vertices of the auxiliary lattice and* $\det \boldsymbol{J}$ *means the determinant of* $\boldsymbol{J}$ *over all* N^{Δ}*-dimensional Hilbert space.*

The reader may find the proof of this proposition in [22]. Equation (28) is indeed the functional equation. $\boldsymbol{J}(\alpha, \beta)$ has a commutative subset of $\Delta - 1$ integrals of motion. In a basis of eigenvectors of this subset $\det \boldsymbol{J}$ contains a product over all $N^{\Delta-1}$ independent eigenvalues of it, corresponding to the $N^{\Delta-1}$th power on the right-hand side of (28). Besides this product, $\det \boldsymbol{J}$ contains an $N \times N$ determinant over the non-commutative mass centre pair, so that this determinant must equal $\mathcal{J}$.

The most visual application of (28) concerns the square $A \times B$ lattice with $N = 2$, $\boldsymbol{u}_v^N = \boldsymbol{w}_v^N = 1$ and $\kappa_v = q^{-1/2}$. This case corresponds to the layer-to-layer transfer matrix of Zamolodchikov's model in a special regime. Eigenvalues of the commutative subset of $\boldsymbol{J}$ obey the functional equation:

$$j(\alpha, \beta) j(-\alpha, -\beta) + j(-\alpha, \beta) j(\alpha, -\beta) = 2(1 - \alpha^2)^A (1 - \beta^2)^B \tag{29}$$

where the eigenvalue of the auxiliary transfer matrix $j(\alpha, \beta)$ is a polynomial of α, β.

It is interesting to discuss a little the role of the functional part of $\boldsymbol{R}$ mapping on Nth powers in the application to non-evolving lattices (square or spiral, for example). On such lattices there are vertices of only one type, and the Legendre transformation such as (17) implies only two types of τ functions, depending on a two-dimensional spatial discrete coordinate. 'Equations of motion' for these τ functions come from the isospectrality problem for $\mathcal{J}$ (and consequently, for $\boldsymbol{J}$), and their solutions are the two-dimensional solitons. The functional part of the quantum transfer matrix (besides the trivial one) is the vertex operator, making the Bäcklund transformation (all these are discussed in [24]). Amusingly, the finite-dimensional parts of the nontrivial Q operators give explicitly several constructions for the separation of variables method. This phenomenon in its simplest case is the subject of a future paper [23].

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